2501. (a) The possibility space is the set of six pairs:

$$\{OA, OB, OC, AB, AC, BC\}.$$

Four of these (the sides of the square) yield a distance of 1. So, $p = 2/3$.

(b) The possibility space is reduced to $\{OB, AC\}$, so $p = 1/2$.

2502. By the chain rule, the derivative is

$$\frac{dy}{dx} = -xe^{-\frac{1}{2}x^2}.$$

At $x = -1$, both the y value and the gradient are $e^{-\frac{1}{2}}$. So, the equation of the tangent is

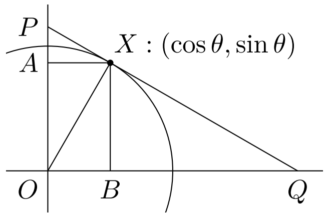
$$y - e^{-\frac{1}{2}} = e^{-\frac{1}{2}}(x + 1).$$

Setting $y = 0$, we can multiply by $e^{\frac{1}{2}}$, which leaves $-1 = x + 1 \implies x = -2$, as required.

2503. We expand and simplify as follows:

$$\begin{aligned}
&\lim_{k\to 0} \frac{(6+k)^3 - 6^3}{(6+k)^2 - 6^2} \\
&= \lim_{k\to 0} \frac{k^3 + 18k^2 + 108k}{k^2 + 12k} \\
&= \lim_{k\to 0} \frac{k^2 + 18k + 108}{k + 12} \\
&= \frac{108}{12} \\
&= 9.
\end{aligned}$$

2504. Dropping perpendiculars to the axes, we have



$\triangle OBX$ is similar to $\triangle OXQ$. Hence,

$$\frac{|OQ|}{|OX|} = \frac{|OX|}{|OB|}.$$

We also know that $|OB| = \cos\theta$ and $|OX| = 1$, so $|OQ| = \sec\theta$. Since sin and cos are symmetrical in $y = x$, this means $|OP| = \operatorname{cosec}\theta$, as required.

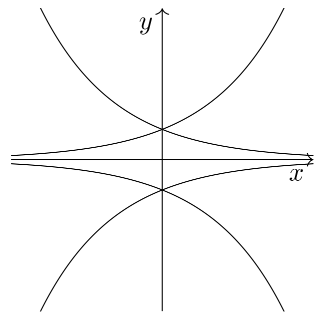
2505. The three curves are $y = e^x$

(a) reflected in the y axis,

(b) reflected in the x axis,

(c) reflected in both the x and y axes (equivalently rotated 180° around O).

So, the graphs are as follows:



The answers are

(a) Yes,

(b) No,

(c) No.

2506. The two quadratic inequalities are

$$x^2 \pm x - 6 < 0.$$

These have boundary equations

$$x^2 \pm x - 6 = 0.$$

The roots are $x = -3, 2$ and $x = -2, 3$. So, we can simplify to $(-3, 2) \cap (-2, 3)$, which is $(-2, 2)$.

2507. The quotient rule gives

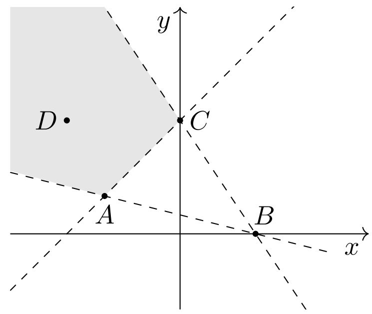
$$\begin{aligned}
\frac{dy}{dx} &= \frac{\left((\ln x)^2\right)'(2 - \ln x) - (\ln x)^2 (2 - \ln x)'}{(2 - \ln x)^2} \\
&= \frac{2\ln x \cdot \frac{1}{x}(2 - \ln x) - (\ln x)^2 \cdot -\frac{1}{x}}{(2 - \ln x)^2} \\
&= \frac{4\ln x - (\ln x)^2}{x(2 - \ln x)^2}.
\end{aligned}$$

For stationary points,

$$\begin{aligned}
& 4\ln x - (\ln x)^2 = 0 \\
\implies & \ln x(4 - \ln x) = 0 \\
\implies & \ln x = 0, 4 \\
\implies & x = 1, e^4.
\end{aligned}$$

So the stationary points are $(1, 0)$ and $(e^4, -8)$.

2508. The scenario is as follows:



In order for $ABCD$ to be convex, D must be in the region shaded above. Finding the boundary lines, we require $3x + 2y < 6$, $x + 4y > 2$ and $y > x + 3$.

2509. Differentiating by the product and chain rules,

$$\begin{aligned}&\frac{d}{dt}(x^2y^2)\\ &\equiv \left(\frac{d}{dt}\left(x^2\right)\right)y^2 + x^2\left(\frac{d}{dt}\left(y^2\right)\right)\\ &\equiv 2x\frac{dx}{dt}y^2 + x^2\cdot 2y\frac{dy}{dt}\\ &\equiv 2xy(ay+bx).\end{aligned}$$

2510. Differentiating from first principles,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{k^{x+h}-k^x}{h}\\ &\equiv \lim_{h\to 0}\frac{k^xk^h-k^x}{h}\\ &\equiv \lim_{h\to 0}\frac{k^x(k^h-1)}{h}.\end{aligned}$$

We can take the factor of k^x out of the limit, as it doesn't depend on h. Equating the derivative to k^x (as given in the question),

$$k^x\lim_{h\to 0}\frac{k^h-1}{h} = k^x.$$

Since $k^x \neq 0$, we can divide through by it, giving

$$\lim_{h\to 0}\frac{k^h-1}{h} = 1, \text{ as required.}$$

2511. (a) In harmonic form $R\sin(\theta+\alpha)$, the amplitude R is the Pythagorean sum of the amplitudes of the individual sinusoids. So,

$$R = \sqrt{7^2+24^2} = 25.$$

Hence, the range of the function is $[-15°, 35°]$.

(b) The temperature oscillates with period 2π. So, ignoring the phase shift (any translation in the t direction associated with α will have no effect on the probability), we consider one period of $T = 25\sin t + 10$, and look for $T > 22.5$:

$$\begin{aligned}&25\sin t + 10 = 22.5\\ \Longrightarrow\ &25\sin t = 12.5\\ \Longrightarrow\ &t = \tfrac{\pi}{6}\pi, \tfrac{5\pi}{6}.\end{aligned}$$

So, the system is warm when $t \in (\pi/6, 5\pi/6)$, an interval of length $2\pi/3$. This gives

$$\mathbb{P}(\text{system warm}) = \frac{2\pi/3}{2\pi} = \frac{1}{3}.$$

2512. Using the product rule and standard derivatives,

$$\begin{aligned}&\frac{d}{d\theta}(\sec\theta\operatorname{cosec}\theta)\\ &\equiv \sec x\tan x\cdot\operatorname{cosec}x + \sec x\cdot -\operatorname{cosec}x\cot x\\ &\equiv \sec x\frac{\sin x}{\cos x}\operatorname{cosec}x - \sec x\operatorname{cosec}x\frac{\cos x}{\sin x}\\ &\equiv \sec^2 x - \operatorname{cosec}^2 x, \text{ as required.}\end{aligned}$$

2513. There are three possibilities here:

① more heads than tails,
② n heads and n tails,
③ more tails than heads.

The distribution $B(2n, 1/2)$ is symmetrical, so the first and third possibilities are equally likely. The second has probability

$$\mathbb{P}(n \text{ heads}, n \text{ tails}) = {}^{2n}\mathrm{C}_n(1/2)^{2n}.$$

Subtracting this from the total probability 1, we then divide by two:

$$\begin{aligned}&\mathbb{P}(\text{more heads than tails})\\ &= \tfrac{1}{2}\left(1 - {}^{2n}\mathrm{C}_n\tfrac{1}{2}^{2n}\right)\\ &\equiv \tfrac{1}{2}\left(1 - \frac{{}^{2n}\mathrm{C}_n}{2^{2n}}\right), \text{ as required.}\end{aligned}$$

2514. (a) The curve $y = ax^2 + bx + c$ has derivative

$$\frac{dy}{dx} = 2ax + b.$$

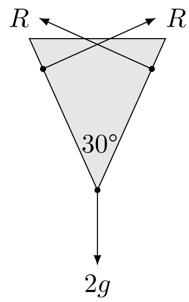
Substituting in, we require

$$\begin{aligned}&(2ax+b)^2 - (ax^2+bx+c) \equiv 0\\ \Longrightarrow\ &(4a^2-a)x^2 + (4ab-b)x + b^2 - c \equiv 0.\end{aligned}$$

Coefficients of x^2 are $4a^2 - a = 0$. Since $a \neq 0$, this gives $a = \frac{1}{4}$.

(b) The point $(2, 0)$ requires $0 = 1+2b+c$, and the point $(-2, 4)$ requires $4 = 1 - 2b + c$. Solving simultaneously gives $b = -1$, $c = 1$. So, the solution curve is $y = \frac{1}{4}x^2 - x + 1$.

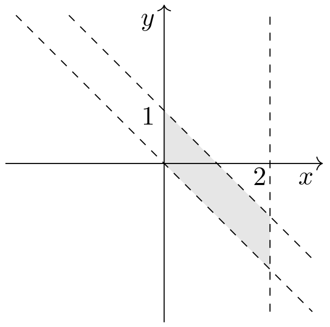
2515. (a) The force diagram is



Resolving vertically, $2R\sin 15° - 2g = 0$, so $R = 37.9$ N (3sf).

(b) The cone will fall with an initial acceleration a which is slightly less than g. This is because, until the cone has fallen a short distance, the reaction from the second support will still have a vertical component.

2516. The boundary equations are $x = 0, 2$ for the first inequality and $x + y = 0, 1$ for the second. So, R is bounded by two pairs of parallel lines:



The side $x = 0$ has length 1 and the perpendicular "height" (in the x direction) is 2. Hence, R is a parallelogram with area $1 \times 2 = 2$, as required.

2517. We rearrange and integrate as follows:

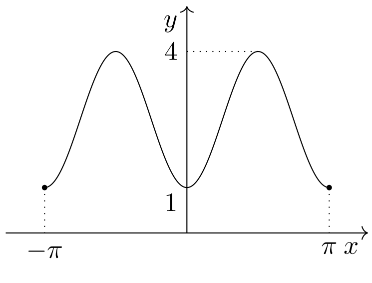
$$\begin{aligned} \int \frac{1}{y}\,dy &= -\int \frac{1}{x}\,dx \\ \Longrightarrow \ln|y| &= -\ln|x| + c \\ \Longrightarrow \ln|y| &= \ln\left|\tfrac{1}{x}\right| + c \\ \Longrightarrow |y| &= \left|\tfrac{1}{x}\right| e^c. \end{aligned}$$

This allows all reciprocal graphs, with variables x and y either positive or negative. As it stands, e^c is positive, but we can describe all solution curves by changing the positive e^c for any constant A, and getting rid of the mod signs. The general solution is all reciprocal graphs: $y = \frac{A}{x}$.

2518. We can rearrange $\cos 2x \equiv 1 - 2\sin^2 x$ to the form $\sin^2 x \equiv \frac{1}{2} - \frac{1}{2}\cos 2x$. Substituting in, the graph is

$$\begin{aligned} y &= 5\sin^2 x + \cos 2x \\ &\equiv \tfrac{5}{2} - \tfrac{5}{2}\cos 2x + \cos 2x \\ &\equiv \tfrac{5}{2} - \tfrac{3}{2}\cos 2x. \end{aligned}$$

This is $y = \cos x$ transformed by: ① a stretch by factor $1/2$ in the x direction, ② a stretch/reflection factor $-3/2$ in the y direction, and ③ translation by vector $5/2\mathbf{j}$. This gives



2519. (a) $Z_1 \sim N(0, 1)$. So, using a calculator's normal facility, $\mathbb{P}(|Z_1| < 1) = 0.683$ (3sf).

(b) $\bar{Z} \sim N(0, 1/5)$. So, using a calculator's normal facility, $\mathbb{P}(|\bar{Z}| < 1) = 0.975$ (3sf).

(c) The situation is symmetrical: Z_1 is equally likely to be above or below the mean of the sample of five. So, $\mathbb{P}(Z_1 < \bar{Z}) = 0.5$.

2520. To return to point A, the ant must circumnavigate one of the faces. It makes no difference in which direction the ant goes first, as the octahedron is symmetrical. The second leg must be either left or right, but not straight ahead, so $p_2 = 2/3$. Only one choice for the third leg will complete the triangle, so $p_3 = 1/3$. Hence, the probability is $p_2 p_3 = 2/9$.

2521. We need to rename one of the t's, as the t-value will differ line by line at the point of intersection. The intersection satisfies

$$\begin{aligned} 1 - 4s &= 3 + t, \\ 2 - 2s &= -3 - t. \end{aligned}$$

Solving, $s = 1/2$ and $t = -4$, so the point is $(-1, 1)$. Substituting this into the LHS of the inequality, $(-1-1)^2 + 2 \cdot 1^2 = 6 < 7$. Hence, the lines meet in the elliptical region, as required.

2522. Set $y = x - 1$, and rewrite as $x = y + 1$. This gives

$$\begin{aligned} &x^6 - 6x^4 \\ &= (y+1)^6 - 6(y+1)^4 \\ &\equiv y^6 + 6y^5 + 15y^4 + 20y^3 + 15y^2 + 6y + 1 \\ &\qquad + 6(y^4 + 4y^3 + 6y^2 + 4y + 1) \\ &\equiv y^6 + 6y^5 + 21y^4 + 44y^3 + 51y^2 + 30y + 7 \\ &= (x-1)^6 + 6(x-1)^5 + 21(x-1)^4 + 44(x-1)^3 \\ &\qquad + 51(x-1)^2 + 30(x-1) + 7. \end{aligned}$$

2523. Let the initial velocity be u at an angle θ above horizontal. Components are $u\cos\theta$ horizontally and $u\sin\theta$ vertically. So, $v^2 = u^2 + 2as$ vertically gives $v_y^2 = u^2\sin^2\theta + 2gh$. The final horizontal velocity is $v_x = u\cos\theta$. The overall final speed is given, then, by

$$\begin{aligned} v &= \sqrt{v_x^2 + v_y^2} \\ &= \sqrt{u^2\cos^2\theta + u^2\sin^2\theta + 2gh} \\ &\equiv \sqrt{u^2 + 2gh}. \end{aligned}$$

This doesn't depend on θ, so landing speed is the same whatever the angle of projection.

2524. Reflections and rotations maintain the shape of the parabola: they do not stretch it. But the leading coefficients of the two parabolae are 3 and -2. The minus sign can be generated by either a rotation or a reflection, but the scale factor $2/3$ cannot.

2525. $\mathbb{R} \setminus \mathbb{Z}$ is the set of all non-integer real numbers; $\mathbb{R} \setminus \mathbb{Q}$ is the set of all irrational real numbers. E.g. $1/2$ is in the former but not the latter. In the other direction, all irrationals are non-integers. Hence, the implication is $x \in \mathbb{R} \setminus \mathbb{Z} \impliedby x \in \mathbb{R} \setminus \mathbb{Q}$.

2526. Differentiating from first principles, we multiply top and bottom by the inlaid denominators:

$$\begin{aligned}\frac{d}{dx}\left(x^{-\frac{1}{2}}\right) &\equiv \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}\\ &\equiv \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}.\end{aligned}$$

We now use the technique for rationalising surds, which is multiplication by a conjugate. This gives

$$\lim_{h\to 0}\frac{\left(\sqrt{x}-\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}.$$

Expanding the numerator, this is

$$\begin{aligned}&\lim_{h\to 0}\frac{-h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}\\ \equiv\ &\lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}\\ \equiv\ &\frac{-1}{2x\sqrt{x}}\\ \equiv\ &-\tfrac{1}{2}x^{-\frac{3}{2}}, \text{ as required.}\end{aligned}$$

2527. (a) We replace y by $-y$, giving $\mathrm{f}(x)+\mathrm{g}(-y)=0$,
(b) We switch x and y, giving $\mathrm{f}(y)+\mathrm{g}(x)=0$.

2528. We use $\cot^2 x+1\equiv\operatorname{cosec}^2 x$. This gives

$$\begin{aligned}&\cot^2\tfrac{3\pi}{8}=\left(4-2\sqrt{2}\right)-1=3-2\sqrt{2}\\ \therefore\quad &\cot\tfrac{3\pi}{8}=\sqrt{3-2\sqrt{2}}\\ \Longrightarrow\ &\tan\tfrac{3\pi}{8}=\frac{1}{\sqrt{3-2\sqrt{2}}}.\end{aligned}$$

2529. For the ith data set, the sum of the data in that set is given by $n_i\bar{x}_i$. To find $\bar{x}_{\text{total}}$, we add these and divide by the total number of data:

$$\bar{x}_{\text{total}}=\frac{\displaystyle\sum_{i=1}^{k}n_i\bar{x}_i}{\displaystyle\sum_{i=1}^{k}n_i}.$$

2530. Completing the square for x and y, the circles are $(x-3)^2+y^2=4$ and $(x+2)^2+(y+2)^2=11$. These have centres $(3,0)$ and $(-2,-2)$ and radii 2 and $\sqrt{11}$. The distance between the centres is $\sqrt{5^2+2^2}=\sqrt{29}\approx 5.385$. The sum of the radii is $2+\sqrt{11}\approx 5.317$. Since the latter is slightly smaller than the former, the circles do not intersect, but do approach each other closely.

2531. Begin with $a\le b$, so $\min(a,b)=a$. This gives

$$x+y>k \implies ax+ay>ak.$$

And, since $b\ge a$,

$$ax+by\ge ax+ay>ak=k\min(a,b).$$

The same argument holds if $a\ge b$. QED.

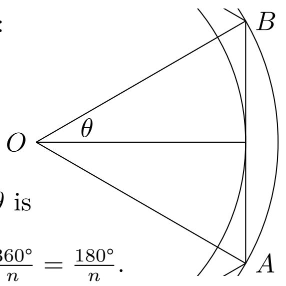
2532. Each second, 5 kg of exhaust is accelerated to 400 ms^{-1}. The acceleration of the exhaust is therefore 400 ms^{-2}. So, the force exerted on the exhaust is a constant $F=ma=5\times 400=2000$ N. By NIII, the same force must be exerted on the engine, so the initial acceleration is 10 ms^{-2}.

— Nota Bene —

If each kg of exhaust took a time other than 1 s to accelerate, the answer would be the same. Let it take t seconds. The acceleration of the exhaust is $\frac{400}{t}$ ms^{-2}. Since 5 kg is accelerated every second, the mass accelerated in t seconds is $5t$ kg. Now, $F=ma=\frac{400}{t}\times 5t=2000$ N, as before.

2533. This isn't true. Consider a possibility space of four equally likely outcomes, and events $A=\{1,2\}$, $B=\{1,3\}$ and $C=\{3,4\}$. Events A and B are independent, as are B and C. A and C, however, are mutually exclusive, therefore dependent.

2534. The scenario is:



(a) The angle θ is

$$\theta=\tfrac{1}{2}\cdot\tfrac{360^\circ}{n}=\tfrac{180^\circ}{n}.$$

The ratio of the radii is then $1:\cos\frac{180^\circ}{n}$. Since the circumference scales with the radius, this is also the ratio of circumferences.

(b) As $n\to\infty$, $\frac{180^\circ}{n}\to 0$. So, since $\cos 0=1$, the ratio of the circumferences tends to 1.

2535. This is a quadratic in x. Using the formula,

$$x=\frac{-qyz\pm\sqrt{q^2y^2z^2-4pry^2z^2}}{2p}.$$

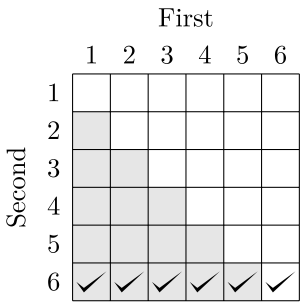
2536. Square rooting to $z=\pm\sqrt{4x+1}$, we integrate by the reverse chain rule:

$$\begin{aligned}&\int\frac{2}{z}\,dx\\ =\ &\int\pm 2(4x+1)^{-\frac{1}{2}}\,dx\\ \equiv\ &\pm 2\cdot 2\cdot\tfrac{1}{4}(4x+1)^{\frac{1}{2}}+c\\ =\ &z+c, \text{ as required.}\end{aligned}$$

— Nota Bene —

This problem looks like it it needs integration by substitution. But, since the integral of $\frac{2}{z}$ is *with respect to* x (not z), there is no need.

2537. The possibility space is as follows:



With the possibility space restricted to the shaded region, the probability that the second die was a six is $^5/_{15} = {}^1/_3$.

2538. (a) Writing in column vectors,

$$\phi(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad \dot{\phi}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

The gradients are negative reciprocals, so these vectors are perpendicular.

(b) The second derivative is $\ddot{\phi}(t) = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}$.

(c) Considering $\phi(t)$ as a position vector at time t, this is circular motion at constant speed. And from (b),

$$\ddot{\phi}(t) = -\phi(t).$$

The acceleration vector is the negative of the position vector, i.e. back towards the centre. This is the *centripetal acceleration* of circular motion.

2539. (a) Since 6 is even, a positive sextic must have a local (and indeed global) minimum.

(b) Neither $y = x^5$ nor $y = x^3$ has a minimum, so their sum doesn't either.

(c) Since 4 is even, a positive quartic must have a local (and indeed global) minimum.

2540. (a) The five relevant points are

x	0	0.25	0.5	0.75	1
y	0	0.0625	0.25	0.5625	1

By Pythagoras, the sum of distances between successive points is

$$\frac{\sqrt{17}}{16} + \frac{5}{16} + \frac{\sqrt{41}}{16} + \frac{\sqrt{65}}{16} = 1.47428...$$

The percentage error is then

$$\frac{1.47428 - l}{l} = -0.0031528...$$

So, the straight lines underestimate the true arc length by 0.32% (2sf).

(b) A straight line is the shortest distance between two points, so approximating any non-straight curve with straight lines must underestimate arc length.

2541. Substituting the second equation into the first, we use log rules to solve:

$$\begin{aligned} & \log_2 x + 2\log_4(6-x) = 3 \\ \Longrightarrow\ & \log_4 x^2 + \log_4 (6-x)^2 = 3 \\ \Longrightarrow\ & \log_4 \left(x^2(6-x)^2\right) = 3 \\ \Longrightarrow\ & x^2(6-x)^2 = 64 \\ \Longrightarrow\ & x^4 - 12x^3 + 36x^2 - 64 = 0. \end{aligned}$$

Using a polynomial solver, $x = 2, 4, 3 \pm \sqrt{17}$. The surds don't satisfy the original equations, as $3 - \sqrt{17} < 0$ and $6 - \left(3 + \sqrt{17}\right) < 0$. Hence, the intersections are $(2, 4)$ and $(4, 2)$.

2542. The derivative is $2x$, so the gradient at $\left(a, a^2\right)$ is $2a$. This means that the angle of elevation θ of the tangent satisfies $\tan\theta = 2a$, which gives $\theta = \arctan 2a$.

2543. (a) This is false: $\dfrac{1}{a} + \dfrac{1}{b} \not\equiv \dfrac{1}{a+b}$.

(b) This is true: summation is linear.

(c) This is false: $ab + cd \not\equiv (a+b)(c+d)$.

2544. (a) The derivative is

$$\frac{dy}{dx} = \tfrac{3}{4}x^{-\frac{1}{4}} - x^{-\frac{1}{2}} + \tfrac{1}{4}x^{-\frac{3}{4}}.$$

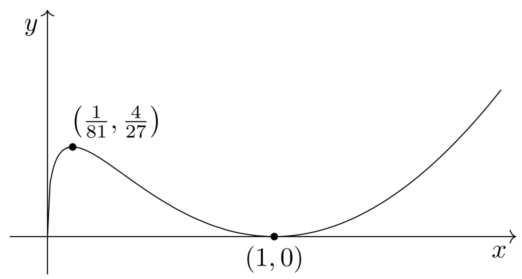
Each term is undefined at $x = 0$, which means the gradient is infinite at the origin. Hence, the tangent is $x = 0$.

(b) Solving for stationary points,

$$\begin{aligned} & \tfrac{3}{4}x^{-\frac{1}{4}} - x^{-\frac{1}{2}} + \tfrac{1}{4}x^{-\frac{3}{4}} = 0 \\ \Longrightarrow\ & 3x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 1 = 0 \\ \Longrightarrow\ & (3x^{\frac{1}{4}} - 1)(x^{\frac{1}{4}} - 1) = 0 \\ \Longrightarrow\ & x^{\frac{1}{4}} = 1, \tfrac{1}{3} \\ \Longrightarrow\ & x = 1, \tfrac{1}{81}. \end{aligned}$$

So, there are stationary points at $(1, 0)$ and $(^1/_{81}, {}^4/_{27})$. The second derivative is positive at $(1, 0)$ and negative at $x = {}^1/_{81}$, so these are a local minimum and maximum respectively.

(c) As $x \to \infty$, $y \to \infty$. Combining this fact with that of (a) and (b), a sketch of the graph is



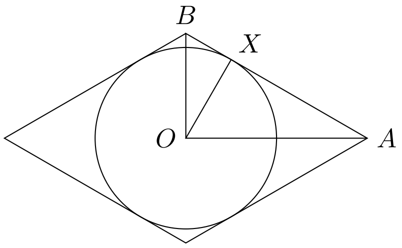
2545. Multiplying up by the denominator,

$$\begin{aligned} & x^2 + \frac{x-1}{x+1} = 3 - 2x \\ \Longrightarrow\ & x^2(x+1) + x - 1 = (3-2x)(x+1) \\ \Longrightarrow\ & x^3 + 3x^2 - 4 = 0. \end{aligned}$$

We spot that $x = 1$ is a root, which means $(x-1)$ is a factor. This gives

$$\begin{aligned} & (x-1)(x^2+4x+4) = 0 \\ \Longrightarrow\ & (x-1)(x+2)^2 = 0 \\ \Longrightarrow\ & x = 1, -2. \end{aligned}$$

2546. We can split the rhombus into four right-angled triangles with sides $\left(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}\sqrt{a^2+b^2}\right)$:



Then $\triangle OAX$ is similar to $\triangle OBX$. Hence, with $\frac{1}{2}b$ taken to be opposite B, i.e. $|OA|$,

$$\begin{aligned} & \frac{|OX|}{|OA|} = \frac{|OB|}{|AB|} \\ \Longrightarrow\ & \frac{|OX|}{\frac{1}{2}b} = \frac{\frac{1}{2}a}{\frac{1}{2}\sqrt{a^2+b^2}} \\ \Longrightarrow\ & |OX| = \frac{ab}{2\sqrt{a^2+b^2}}, \text{ as required.} \end{aligned}$$

2547. We simplify $\mathrm{f}\big(\mathrm{f}(x)\big)$ as follows, multiplying up by the inlaid denominators:

$$\begin{aligned} \mathrm{f}^2(x) &= \frac{\frac{x+2}{2x+1} + 2}{2\frac{x+2}{2x+1} + 1} \\ &\equiv \frac{x+2+2(2x+1)}{2(x+2)+2x+1} \\ &\equiv \frac{5x+4}{4x+5}. \end{aligned}$$

2548. Tangents parallel to y for the inverse trig graphs are reflections of tangents parallel to x for the trig graphs. Over the relevant restricted domains, these are at $(-\pi/2, -1)$ and $(\pi/2, 1)$ for sin, $(0, 1)$ and $(\pi, -1)$ for cos, and there are none for tan. On reflection in $y = x$, this gives tangents parallel to y at

(a) $(-1, -\pi/2)$ and $(1, \pi/2)$ for $y = \arcsin x$,

(b) $(1, 0)$ and $(-1, \pi)$ for $y = \arccos x$,

(c) None for $y = \arctan x$.

2549. Factorising, $3x^2 + 7xy + 2y^2 = 0$ can be expressed as $(3x+y)(x+2y) = 0$. This holds iff $3x + y = 0$ or $x + 2y = 0$. Each of these is a straight line through the origin. Hence, the locus is a pair of intersecting lines, as required.

2550. Call the first couple A_1, A_2. The probability that A_2 sits next to A_1 is $2/7$. Then someone must sit next to A_1; call them B_1. The probability that B_2 then sits next to B_1 is $1/5$. Call B_2's neighbour C_1. The probability that C_2 sits next to C_1 is $1/3$. Then place D_1 and D_2; they certainly sit next to each other. So, the probability is

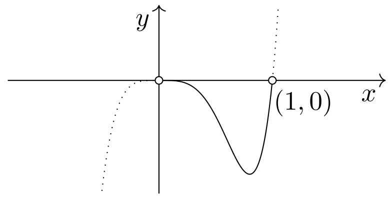
$$p = \tfrac{2}{7} \times \tfrac{1}{5} \times \tfrac{1}{3} = \tfrac{2}{105}, \text{ as required.}$$

— Alternative Method —

There are 8! ways in which the people can sit down. The number of successful outcomes is as follows. There are 2 choices for the locations of couples. Then there are 4! orders of the couples. Then each couple can be ordered in 2! ways. So,

$$p = \frac{2 \times 4! \times (2!)^4}{8!} = \tfrac{2}{105}, \text{ as required.}$$

2551. The graph $y = x^5 - x^4$ is as follows:



The denominator $x^5 - x^4$ has roots at $x = 0$ and $x = 1$, but is negative on the domain $(0, 1)$, which excludes 0 and 1. So, the function h is well defined.

2552. Multiplying up, we require

$$1 \equiv A(x^2-1) + Bx^2(x-1) + Cx^2(x+1).$$

Equating coefficients,

$$\begin{aligned} x^3 &: 0 = B + C, \\ x^2 &: 0 = A - B + C, \\ x^0 &: 1 = -A. \end{aligned}$$

So, $A = -1$, $B = -\frac{1}{2}$ and $C = \frac{1}{2}$.

2553. (a) The initial period lasted for $\frac{9}{a}$ seconds. So, the distance covered by A was

$$\tfrac{1}{2} \cdot 9 \cdot \tfrac{9}{a} + 9\big(11.5 - \tfrac{9}{a}\big) = 100.$$

Solving this, $a = 11.6 \text{ ms}^{-2}$ (3sf).

(b) Let maximum speed be at $(t, 11)$. Then the distance covered is

$$\tfrac{1}{2} \cdot 11t + \tfrac{1}{2}(11+7)(11.5-t) = 96.$$

Solving this, $t = 2.14$ s (3sf).

2554. (a) Log rules give $\ln {x^2}/{y} = k$. Exponentiating both sides,

$$\begin{aligned} \frac{x^2}{y} &= e^k \\ \Longrightarrow ye^k &= x^2. \end{aligned}$$

Hence, if $k = 0$, then $e^k = 1$ and we have the parabola $y = x^2$. But $\ln x$ is undefined for $x < 0$, so this is half of the parabola.

(b) To reflect the half-parabola in the y axis, we replace x by $-x$, giving $2\ln(-x) - \ln y = 0$.

2555. A null hypothesis giving a *set* of possible values for p is no use, as we can't use it to find probabilities. For such a two-tailed test, the hypotheses are

$$\begin{aligned} H_0 &: p = \tfrac{1}{6}, \\ H_1 &: p \neq \tfrac{1}{6}. \end{aligned}$$

This way we can calculate, under the assumption of H_0, the probabilities of various events, and then compare with what occurs in the sample.

2556. With cuboid dimensions $a \times b \times c$,

$$\begin{aligned} &① \; a^2 + b^2 = 125^2, \\ &② \; a^2 + c^2 = 244^2, \\ &③ \; b^2 + c^2 = 267^2. \end{aligned}$$

These are linear equations in the variables a^2, b^2 and c^2. Setting up ① − ② + ③,

$$\begin{aligned} 2b^2 &= 125^2 - 244^2 + 267^2 = 27378 \\ \Longrightarrow b &= 117. \end{aligned}$$

So, the dimensions are $44 \times 117 \times 240$.

2557. Differentiating implicitly by product/chain rules,

$$\begin{aligned} & x^2y^3 - 2 = xy^{\frac{3}{2}} \\ \Longrightarrow\ & 2xy^3 + 3x^2y^2\frac{dy}{dx} = y^{\frac{3}{2}} + \tfrac{3}{2}xy^{\frac{1}{2}}\frac{dy}{dx} \\ \Longrightarrow\ & \frac{dy}{dx}\left(3x^2y^2 - \tfrac{3}{2}xy^{\frac{1}{2}}\right) = y^{\frac{3}{2}} - 2xy^3 \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{y^{\frac{3}{2}} - 2xy^3}{3x^2y^2 - \frac{3}{2}xy^{\frac{1}{2}}}. \end{aligned}$$

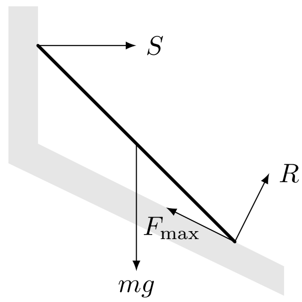
Substituting $(-1, 1)$, the gradient is $m = {2}/{3}$. So, the equation of the tangent is

$$\begin{aligned} y - 1 &= \tfrac{2}{3}(x + 1) \\ \Longrightarrow 3y &= 2x + 5. \end{aligned}$$

2558. Rearranging, we have $\cos t = \frac{x}{2}$ and $\sin t = \frac{y-5}{3}$. Squaring and adding the equations, we then use $\cos^2 t + \sin^2 t \equiv 1$, which gives

$$\frac{x^2}{4} + \frac{(y-5)^2}{9} = 1.$$

2559. (a) Forces:



The line of action of S is twice as far from the foot of the plank as the line of action of the weight is. So, taking moments around the foot, $S = \frac{1}{2}mg$.

(b) Resolving vertically,

$$R\cos 30° + F_{\max}\sin 30° - mg = 0.$$

Since the plank is on the point of slipping, we set $F_{\max} = \mu R$. Multiplying by two and then rearranging, $R(\sqrt{3} + \mu) = 2mg$.
Resolving horizontally,

$$R\sin 30° + \tfrac{1}{2}mg - F_{\max}\cos 30° = 0.$$

Again we set $F_{\max} = \mu R$, multiply by two and rearrange. This gives $R(\sqrt{3}\mu - 1) = mg$.

(c) Dividing the two equations above, the plank is on the point of slipping if

$$\begin{aligned} & \frac{\sqrt{3} + \mu}{\sqrt{3}\mu - 1} = 2 \\ \Longrightarrow\ & \sqrt{3} + \mu = 2\sqrt{3}\mu - 2 \\ \Longrightarrow\ & 2 + \sqrt{3} + \mu = (2\sqrt{3} - 1)\mu \\ \Longrightarrow\ & \mu = \frac{2 + \sqrt{3}}{2\sqrt{3} - 1}. \end{aligned}$$

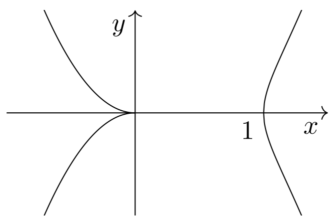
The plank will remain in equilibrium if μ is greater than or equal to this limiting value. Hence,

$$\mu \geq \frac{2 + \sqrt{3}}{2\sqrt{3} - 1}.$$

2560. At x axis intercepts, $y = 0$. So, factorising $x^4 - x^3$, the equation for x intercepts is $x^3(x - 1) = 0$. This has a triple root at $x = 0$, which is therefore a point of tangency. Hence, the curve $y^2 = x^4 - x^3$ is tangent to the x axis at the origin.

——— Nota Bene ———

The curve is only defined one side of $x = 0$, with no points for which $x \in (0, 1)$. However, this doesn't stop a definition of tangency at the x axis.



2561. All four are true.

The cubic is positive, with a single root at $-p < 0$ and a double root at $p > 0$, so must be of the form

$$y = a(x+p)(x-p)^2,$$

where $a, p > 0$. Expanding, this is

$$y = ax^3 - apx^2 - ap^2x + ap^3.$$

(a) a is positive, because the cubic is positive.
(b) $b = -ap$ is negative,
(c) $c = -ap^2$ is negative,
(d) $d = ap^3$ is positive.

2562. We have $u_1 = a$, so $u_2 = \sqrt{a^2 + k}$. This gives

$$\begin{aligned} u_3 &= \sqrt{u_2^2 + k} \\ &\equiv \sqrt{a^2 + k + k} \\ &\equiv \sqrt{a^2 + 2}. \end{aligned}$$

Continuing in the same vein,

$$\begin{aligned} u_4 &= \sqrt{u_3^2 + k} \\ &\equiv \sqrt{a^2 + 2k + k} \\ &\equiv \sqrt{a^2 + 3k}. \end{aligned}$$

This pattern continues, giving nth term

$$u_{n+1} = \sqrt{a^2 + nk}.$$

2563. Expressing in terms of sin and cos,

$$\begin{aligned} &\frac{1}{1+\tan x} + \frac{1}{1+\cot x} \\ \equiv\ &\frac{1}{1+\frac{\sin x}{\cos x}} + \frac{1}{1+\frac{\cos x}{\sin x}}. \end{aligned}$$

Multiplying top and bottom of each term by the denominator of its inlaid fraction, this is

$$\begin{aligned} &\frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\sin x + \cos x} \\ \equiv\ &\frac{\cos x + \sin x}{\sin x + \cos x} \\ \equiv\ &1, \text{ as required.} \end{aligned}$$

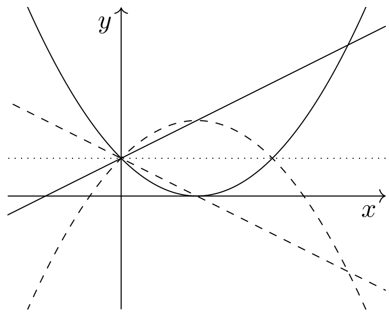
2564. (a) The resultant force is $-8t^2 - 8t - 10$, so the acceleration is $a = -4t^2 - 4t - 5$.

(b) We look for stationary values of a. The rate of change of acceleration is given by

$$\frac{da}{dt} = -8t - 4.$$

Setting this to zero and solving gives $t = -1/2$, which is not in the period $[0, 2]$. Hence, there are no stationary values in the domain. The least value must, therefore, be at either $t = 0$ or $t = 2$. Evaluating, $a = -5$ and $a = -29$ ms^{-2}. The least value of the magnitude of the acceleration is therefore 5 ms^{-2}.

2565. In both cases, the mirror line $y = k$ (shown dotted below) passes through the y intercept. So, the y intercepts stay where they are:



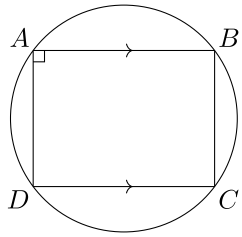
(a) We need a reflected straight line with the same y intercept, so $y = -ax + k$.
(b) In the same way, since the constant term is $+k$, we can output-reflect only the other two terms, giving $y = -ax^2 - bx + k$.

2566. Since $\sum x^2$ is non-negative, so must $\sum x$ and thus $\bar{x}$ be. Assume, for a contradiction, that $\bar{x} > 1$. Now, the sum of squares S_{xx} is also non-negative. So, using the fact that $\sum x = n\bar{x}$, we know that

$$\begin{aligned} &\sum x^2 - n\bar{x}^2 \geq 0 \\ \Longrightarrow\ &\sum x^2 - \bar{x}\sum x \geq 0 \\ \Longrightarrow\ &\sum x^2 \geq \bar{x}\sum x. \end{aligned}$$

But $\bar{x} > 1$, which means that $\sum x^2 > \sum x$. This is a contradiction. Therefore, $\bar{x} \in [0, 1]$.

2567. Let $ABCD$ be a cyclic right-angled trapezium, with AB and CD parallel and a right angle at A.



Opposite angles in a cyclic quadrilateral add to 180°, so angle C is also 90°. And, since AB and CD are parallel, allied angles with A gives D as 90°, and allied angles with C gives B as 90°. Hence, $ABCD$ is a rectangle. □

2568. Integrating with respect to x,

$$\frac{d}{dx}(x+y) = 2x + c.$$

Integrating again,

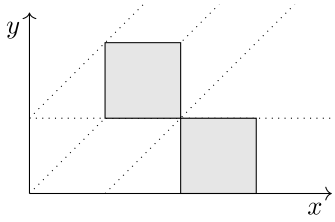
$$\begin{aligned} &x + y = x^2 + cx + d \\ \Longrightarrow\ &y = x^2 + (c-1)x + d. \end{aligned}$$

Redefining constants, this is $y = x^2 + px + q$.

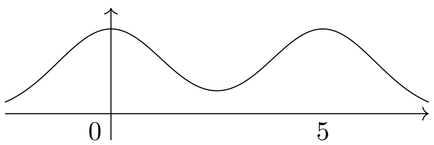
2569. The second square lies on the x axis: its vertices are at $y = 0$ and $y = 1$. The vertices of the first square have y coordinates t and $t + 1$. Hence, the feasible times at which the two squares could share a vertex are $t = -1, 0, 1$. At these times, the lower-left vertices are at

t	Square 1	Square 2
-1	$(-1, -1)$	$(4, 0)$
0	$(0, 0)$	$(3, 0)$
1	$(1, 1)$	$(2, 0)$

At $t = -1, 0$, no vertices are shared. At $t = 1$, both squares have a vertex at $(2, 1)$.



2570. In general, this is not true. Consider $X_1 \sim N(0, 1)$ and $X_2 \sim N(5, 1)$. The combined population is bimodal, with modes at 0 and 5.



This is not a normal distribution.

— Nota Bene —

This remains false even if only the variances differ. It is only true if the sub-population distributions are identical.

2571. Subtracting the two equations, $2\mathbf{b} - \mathbf{c} = 0$, so $2\mathbf{b} = \mathbf{c}$. Since $\mathbf{b}$ and $\mathbf{c}$ are scalar multiples, they must be parallel. And so must e.g. $\mathbf{b}+\mathbf{c}$ be, which tells us, from the first equation, that $\mathbf{a}$ must also be parallel to the other two.

2572. (a) This is a quadratic in x^2. Its discriminant is $\Delta = (a + b)^2 - 4ab$. Multiplying this out, $\Delta = a^2 + b^2 - 2ab \equiv (a - b)^2$, as required.

(b) $\Delta = (a - b)^2$ is non-negative, which means that the quadratic can be solved for x^2. The quadratic formula gives

$$\begin{aligned} x^2 &= \frac{-(a+b) \pm \sqrt{(a-b)^2}}{2} \\ &= \frac{-(a+b) \pm (a-b)}{2} \\ &= -b \text{ or } -a. \end{aligned}$$

But $-b, -a < 0$, so there are no real roots.

2573. For intersections,

$$\begin{aligned} & 3x^3 + 6x^2 - 2 = 2x^2 + 5x \\ \Longrightarrow\ & 3x^3 + 4x^2 - 5x - 2 = 0. \end{aligned}$$

The graphs intersect at $(1, 7)$, so, by the factor theorem, $(x - 1)$ is a factor. This gives

$$\begin{aligned} & (x-1)(3x^2 + 7x + 2) = 0 \\ \Longrightarrow\ & (x-1)(3x+1)(x+2) = 0 \\ \Longrightarrow\ & x = 1, -\tfrac{1}{3}, -2. \end{aligned}$$

Substituting these values back in, the intersections are at $(1, 7)$, $(-1/3, -13/9)$ and $(-2, -2)$.

2574. (a) Equating the first derivatives,

$$\begin{aligned} & 3t^2 - 1 = 2t \\ \Longrightarrow\ & t = 1, -\tfrac{1}{3}. \end{aligned}$$

(b) Equating the moduli of the first derivatives,

$$\begin{aligned} & |3t^2 - 1| = |2t| \\ \Longrightarrow\ & 3t^2 - 1 = \pm 2t \\ \Longrightarrow\ & t = \pm 1, \pm\tfrac{1}{3}. \end{aligned}$$

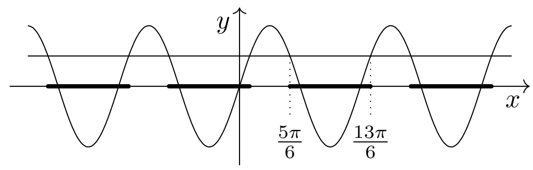
2575. The top is a scalar multiple of the derivative of the bottom, so this can be integrated by inspection:

$$\begin{aligned} & \int_{-1}^{1} \frac{x}{x^2 - 4}\, dx \\ &= \tfrac{1}{2} \int_{-1}^{1} \frac{2x}{x^2 - 4}\, dx \\ &= \tfrac{1}{2} \Big[\ln|x^2 - 4| \Big]_{-1}^{1} \\ &= \tfrac{1}{2}(\ln|-3| - \ln|-3|) \\ &= 0, \text{ as required.} \end{aligned}$$

— Nota Bene —

If further clarification of the inspection is needed, then differentiate $\frac{1}{2}\ln(x^2 - 4)$.

2576. As seen below, the solution set of $\sin x < \frac{1}{2}$ is an infinite series of intervals, each of length $\frac{4}{3}\pi \approx 4.2$:



Each interval shown above, length ≈ 4.2, contains at least 3 integers. And there are infinitely many such intervals. Hence, S contains infinitely many elements. □

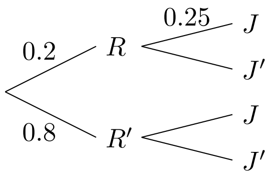
2577. The transformation between the two parabolae is reflection in the x axis. This doesn't affect the roots. And it negates y coordinates, taking (u, v) to $(u, -v)$. Replacing y by $-y$, the equation is

$$-y = px^2 + qx + r$$

— Nota Bene —

Replacing y by $-y$ is equivalent to negating $\mathrm{f}(x)$ (an output transformation). Algebraically, this is $y = -px^2 - qx - r$, the same equation as above.

2578. (a) The probabilities are



Adding the two J branches,

$$\begin{aligned} & 0.2 \times 0.25 + 0.8 \times \mathbb{P}(J \mid R') = 0.1 \\ \Longrightarrow\ & \mathbb{P}(J \mid R') = \tfrac{1}{16}. \end{aligned}$$

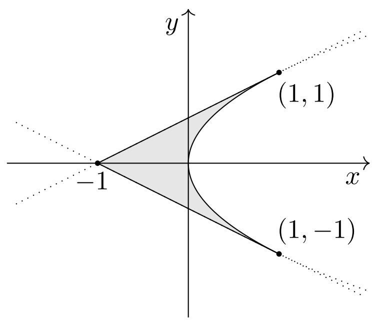
So, $\mathbb{P}(J' \mid R') = \frac{15}{16}$. This gives

$$\begin{aligned} \mathbb{P}(J' \cap R') &= \mathbb{P}(J') \times \mathbb{P}(J' \mid R') \\ &= 0.8 \times \tfrac{15}{16} \\ &= \tfrac{3}{4}. \end{aligned}$$

(b) We restrict the possibility space to the first and third branches:

$$\begin{aligned} \mathbb{P}(R \mid J) &= \frac{\mathbb{P}(R \cap J)}{\mathbb{P}(J)} \\ &= \frac{0.2 \times 0.25}{0.2 \times 0.25 + 0.8 \times \frac{1}{16}} \\ &= \tfrac{1}{2}. \end{aligned}$$

2579. (a) The two lines are tangent to the parabola at $(1, 1)$ and $(1, -1)$, and they cross at $(-1, 0)$:

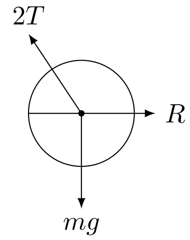


(b) The triangle formed by the three intersections has area 2. Then the region to be subtracted has area

$$2 \times \int_{x=0}^{x=1} \sqrt{x}\, dx = 2 \times \tfrac{2}{3} = \tfrac{4}{3}.$$

So, the area of the logo is $2 - \frac{4}{3} = \frac{2}{3}$.

2580. The force diagram, in cross-section, is



(a) The strings are inclined at 60° to horizontal. Resolving vertically,

$$\begin{aligned} & 2T \sin 60° - mg = 0 \\ \Longrightarrow\ & T = \tfrac{1}{\sqrt{3}} mg. \end{aligned}$$

(b) Resolving horizontally,

$$\begin{aligned} & R - 2T \cos 60° = 0 \\ \therefore\ & R = \tfrac{1}{\sqrt{3}} mg. \end{aligned}$$

This is the force exerted by the cylinder on the wall. By NIII, this has the same magnitude as the force exerted on the wall by the cylinder.

(c) The magnitude of the *total* force included both reaction and tension. Since these, when acting on the cylinder as per NIII, must counteract its weight, the magnitude of the total force on the wall is mg.

2581. (a) Applying the iteration to x once, we get $x^2 - 1$, and twice gives

$$(x^2 - 1)^2 - 1 = x^4 - 2x^2.$$

For period 2, this must return to x, so

$$x^4 - 2x^2 = x.$$

(b) We discard the root $x = 0$, as that gives $0, 0, 0$, which has period 1. This leaves

$$x^3 - 2x - 1 = 0.$$

We spot the root $x = -1$, and factorise to

$$(x + 1)(x^2 - x - 1) = 0.$$

Solving the quadratic,

$$x = -1, \tfrac{1}{2}\big(1 \pm \sqrt{5}\big).$$

2582. Squaring both equations,

$$\begin{aligned} P^2 \operatorname{cosec}^2 \theta &= 289, \\ P^2 \cot^2 \theta &= 64. \end{aligned}$$

The third Pythagorean trigonometric identity is $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$. Subtracting the equations,

$$\begin{aligned} & P^2 \operatorname{cosec}^2 \theta - P^2 \cot^2 \theta = 225 \\ \Longrightarrow\ & P^2 (\operatorname{cosec}^2 \theta - \cot^2 \theta) = 225 \\ \Longrightarrow\ & P^2 \times 1 = 225 \\ \Longrightarrow\ & P = \pm 15. \end{aligned}$$

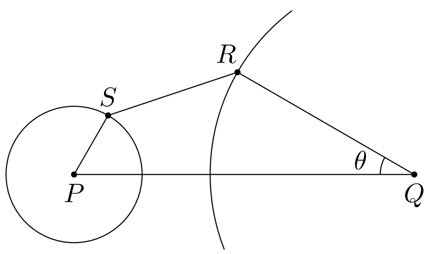
2583. (a) This is a one-tailed test. We define p to be the probability that any roll, in the population of all possible rolls with these dice, gives a six. The hypotheses are

$$\begin{aligned} H_0 &: p = \tfrac{1}{6}, \\ H_1 &: p < \tfrac{1}{6}. \end{aligned}$$

(b) Assuming the null hypothesis, $X \sim B(50, 1/6)$. This gives $\mathbb{P}(X \leq 5) = 0.139$. Since this is greater than 5%, there is insufficient evidence to reject H_0. It seems the inspector's suspicion is unfounded.

(c) A test is only meaningful if different data are used to ① generate and ② test claims. If the same data are used to do both, then the test will be biased towards rejection of H_0, as the effect the test is looking for is already known to exist in the sample.

2584. The shape of $PQRS$ is not fixed by its side lengths: R and S can move on the circles shown.



We are looking for the maximum value of θ, which will occur when PSR is a straight line. This turns $PQRS$ into $\triangle PQR$. The cosine rule gives

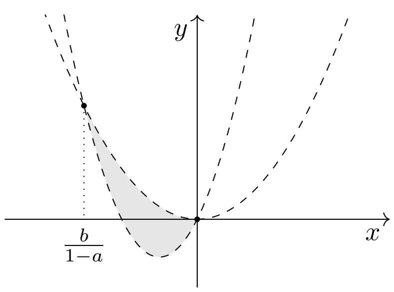
$$\begin{aligned} 3^2 &= 5^2 + 3^2 - 2 \cdot 3 \cdot 5 \cos\theta \\ \implies \cos\theta &= \tfrac{5}{6}. \end{aligned}$$

Since all other cases must produce smaller angles than this, $\angle PQR \leq \arccos\frac{5}{6}$, as required.

2585. For intersections,

$$\begin{aligned} & x^2 = ax^2 + bx \\ \implies & x\big((a-1)x + b\big) = 0 \\ \implies & x = 0, \tfrac{b}{1-a}. \end{aligned}$$

The latter root is negative. So, R is:



The area of R is

$$\begin{aligned} & \int_{\frac{b}{1-a}}^{0} x^2 - (ax^2 + bx)\,dx \\ \equiv & \left[\tfrac{1}{3}(1-a)x^3 - \tfrac{1}{2}bx^2\right]_{\frac{b}{1-a}}^{0}. \end{aligned}$$

Evaluating this gives

$$\begin{aligned} & (0) - \left(\tfrac{1}{3}(1-a)\left(\tfrac{b}{1-a}\right)^3 - \tfrac{1}{2}b\left(\tfrac{b}{1-a}\right)^2\right) \\ \equiv & \left(-\tfrac{1}{3} + \tfrac{1}{2}\right)\frac{b^3}{(1-a)^2} \\ \equiv & \frac{b^3}{6(1-a)^2}, \text{ as required.} \end{aligned}$$

2586. For stationary points,

$$3ax^2 + 2bx + c = 0.$$

If this has real roots, then they are at

$$\begin{aligned} x &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} \\ &\equiv -\frac{b}{3a} \pm \frac{\sqrt{b^2 - 3ac}}{3a} \qquad (*) \end{aligned}$$

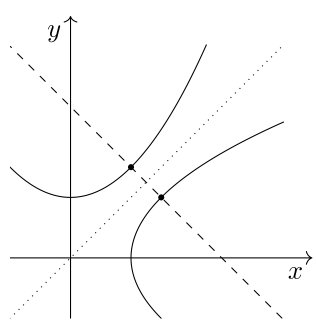
Setting the second derivative to zero for the point of inflection, $6ax + 2b = 0$, which gives $x = -\frac{b}{3a}$. This is the first term of $(*)$, to which a $\pm$ quantity is added. Hence, the two SPs, if they exist, are the same x distance from the point of inflection. □

2587. Factorising, the equations are

$$\begin{aligned} & \big(a\sqrt{b} - 1\big)\big(a\sqrt{b} - 8\big) = 0, \\ & a(a - b) = 0. \end{aligned}$$

The second equation gives $a = 0$ or $a = b$. Subbing $a = 0$ into the first equation produces no values for b. Subbing $a = b$ gives $(1, 1)$ and $(4, 4)$.

2588. The curves are reflections in $y = x$ (dotted). And the shortest distance must lie along the normal (dashed) to both. This has gradient -1, as shown:



Using $2y = x^2 + 2$, the gradient is $\frac{dy}{dx} = x$. We need the normal gradient to be -1, so $x = 1$. Hence, the closest points are $(1, 3/2)$ and $(3/2, 1)$. The distance between these is $\sqrt{2}/2$, as required.

2589. (a) The distributions within the sets differ greatly, with the central half of set A occupying a much broader region than that of B. So, the IQR is much higher for set A, while the range is higher for set B. These are both measures of spread, but offer contradictory conclusions. So neither, on its own, will paint a useful picture of the sample it is trying to summarise.

(b) The mode is likely to lie wherever the density of marks is greatest. This corresponds to the narrowest quarter on the diagram. For set A, this is 75-100%; for set B, it is 25-50%.

2590. A counterexample is $\text{f}(x) = x^2$ and $\text{g}(x) = x^3$. In either order, the composition is $x \mapsto x^{2\times 3}$, but the functions are not identical.

2591. (a) False. At $x = -\frac{\pi}{3}$, the values are

$$\begin{aligned}&\sin\left(-\tfrac{\pi}{3}\right) = -\tfrac{\sqrt{3}}{2},\\ &\cos\left(-\tfrac{\pi}{3}\right) - 1 = -\tfrac{1}{2}.\end{aligned}$$

(b) False. The shape of the graph $y = \tan x$ (the vertical asymptotes) guarantees this.

(c) False, as in (b).

2592. (a) For intersections,

$$\begin{aligned}&\sqrt{x - kx)} + \sqrt{x + kx} = 1\\ \Longrightarrow\ &\sqrt{x}\left(\sqrt{1-k} + \sqrt{1+k}\right) = 1\\ \Longrightarrow\ &\sqrt{x} = \frac{1}{\sqrt{1-k} + \sqrt{1+k}}\end{aligned}$$

The graphs intersect at $x = a$, so

$$\sqrt{a} = \frac{1}{\sqrt{1-k} + \sqrt{1+k}}.$$

(b) The curves intersect at $x = a$, so the RHS of the equation in (a) must be well defined. Hence, each of the square roots in the denominator must be well defined. This requires both of the following to hold:

$$\begin{aligned}1 - k \geq 0 &\Longrightarrow k \leq 1,\\ 1 + k \geq 0 &\Longrightarrow k \geq -1.\end{aligned}$$

So, $|k| \leq 1$, as required.

2593. Combining with a log rule,

$$\begin{aligned}&\log_2\left(2e^x \cdot e^{2x}\right) = 3\\ \Longrightarrow\ &\log_2\left(2e^{3x}\right) = 3\\ \Longrightarrow\ &2e^{3x} = 8\\ \Longrightarrow\ &e^{3x} = 4\\ \Longrightarrow\ &x = \tfrac{\ln 4}{3}.\end{aligned}$$

2594. The centre of the rotational symmetry of a cubic is its point of inflection. Differentiating twice,

$$\begin{aligned}&y = x^3 - 3x^2\\ \Longrightarrow\ &y' = 3x^2 - 6x\\ \Longrightarrow\ &y'' = 6x - 6.\end{aligned}$$

Setting the second derivative to zero gives $x = 1$, so the centre of rotation is $(1, -2)$.

2595. We analyse the relationship (y, z) by integrating with respect to x:

$$\begin{aligned}&\frac{dy}{dx} - \frac{dz}{dx} = 0\\ \Longrightarrow\ &y - z = c\\ \Longrightarrow\ &y = z + c.\end{aligned}$$

We know nothing about the relationships with x.

(a) Considering the case $c \neq 0$, no two variables are necessarily directly proportional.

(b) y and z are necessarily related linearly.

2596. We rearrange and take the cube root:

$$\begin{aligned}&\sin^3 x + 3\sqrt{3}\cos^3 x = 0\\ \Longrightarrow\ &\sin^3 x = -3\sqrt{3}\cos^3 x\\ \Longrightarrow\ &\sin x = -\sqrt{3}\cos x.\end{aligned}$$

Dividing through by $\cos x$,

$$\begin{aligned}&\tan x = -\sqrt{3}\\ \therefore\quad &x = \tfrac{2\pi}{3}, \tfrac{5\pi}{3}.\end{aligned}$$

2597. Differentiating both sides,

$$27\frac{dy}{dx} = 15e^{3x} + 15e^{-3x} - 3.$$

Differentiating both sides again,

$$27\frac{d^2y}{dx^2} = 45e^{3x} - 45e^{-3x}.$$

Substituting into the LHS,

$$\begin{aligned}&\frac{d^2y}{dx^2} - 9y\\ &= \tfrac{1}{27}\left(45e^{3x} - 45e^{-3x} - 9(5e^{3x} - 5e^{-3x} - 3x)\right)\\ &\equiv \tfrac{1}{27}(27x)\\ &\equiv x, \text{ as required.}\end{aligned}$$

2598. This is incorrect. By definition, any force internal to a system must appear as a Newton pair, and the two forces in a Newton pair automatically sum to zero.

2599. The inequality describes the interior of a sphere, centred at $(2, -1, 4)$ with radius 3. The centre is not relevant. The volume of the sphere is

$$V_R = \tfrac{4}{3}\pi r^3 = 36\pi.$$

2600. (a) Differentiating wrt x, we have $\frac{du}{dx} = \cos x$. The reciprocal is then $\frac{dx}{du} = \sec x$.

(b) This is an instance of the chain rule, with $x = \text{f}(u)$ as the inside function and cos as the outside function.

(c) With $u = \sin x$, parts (a) and (b) give

$$\begin{aligned}\frac{d(\cos x)}{d(\sin x)} &= -\sin x \frac{dx}{du} \\ &= -\sin x \sec x \\ &= -\tan x, \text{ as required.}\end{aligned}$$

——— End of 26th Hundred ———